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DISCONTINUOUS VARIATIONAL PROBLEM OF THE OPTIMIZATION OF CONTROL PROCESSES

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V. G. VERETENNIKOV and V. A. SINITSYN
(Moscow)

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On the basis of the results obtained in [1 – 3] we develop a generalized approach to the solution of discontinuous variational problems of the optimization of control processes. We have obtained necessary optimality conditions (the Weierstrass conditions). We have solved the problem of the synthesis of a second-order time-optimal system with constraints of the second type leading to discontinuities in the phase coordinates. We have shown how constraints of the second type, influencing the process to be synthesized, may be taken into account.

1. The systems to be considered are described by ordinary differential equations and relations of the form

$$x_s^{\cdot} = f_s(x, u, t), \quad \varphi_j(u, t) = 0 \quad (s = 1, \ldots, n; \quad j = 1, \ldots; \ m < r) \tag{1.1}$$

Here $x = \{x_1, .., x_n\}$ and $u = \{u_1, ..., u_r\}$ are the phase coordinates and the controls. The functions $x_s(t)$ are continuous and have piecewise-continuous derivatives on the interval $[t_0, T]$ except on a specified number of intervals $[t_i^-, t_i^+] \subset [t_0, T]$ $(i = 0, 1, ..., q)$, on which instead of relations (1.1) there hold the discrepancies

$$\psi_l[x^-(t_0), t_0^-, x^+(t_0), t_0^+, \ldots, x^-(t_q), t_q^-, x^+(t_q), t_q^+] = 0 \tag{1.2}$$
$$(l = 1, \ldots, p < (2n+2)(q+1), \ x^-(t_0) = x(t_0), \ t_0^- = t_0, \ x^+(t_q) = x(T), \ t_q^+ = T)$$

The notation $x^-(t_i)$, $x^+(t_i)$ have the following sense:

$$x^-(t_i) = \lim_{\varepsilon \to 0} x(t_i^- - \varepsilon), \qquad x^+(t_i) = \lim_{\varepsilon \to 0} x(t_i^+ + \varepsilon)$$

The problem is to find, among piecewise-continuous functions $u\ (t)$ $(k = 1, ..., r)$ and $x_s(t)$ $(s = 1, ..., n)$, satisfying (1.1) and (1.2), those which minimize the functional

$$J = \psi_0[x^-(t_0), t_0^-, x^+(t_0), t_0^+, \ldots, x^-(t_q), t_q^-, x^+(t_q), t_q^+] + \sum_{i=0}^{q-1} \int_{t_i^+}^{t_{i+1}} f_0(x, u, t)\, dt$$

Such a formulation serves as a further generalization of the variational problems considered in [1 – 3]. It includes the case when the phase coordinates in the system may have discontinuities of the first kind of a nonfixed magnitude. We also admit discontinuities in the argument t and the possibility that these discontinuities arise at the endpoints of the interval $[t_0, T]$ being studied. We assume that the functions f_s $(s = 0, 1, ..., n)$, φ_j $(j = 1, ..., m)$, ψ_l $(l = 0, 1, ..., p)$ possess the properties mentioned in [4] relative to their arguments. To solve the problem posed we obtain the necessary Weierstrass condition.

2. In the case being examined we can prove the inclusion lemma [4]. The only difference will be in the construction of the admissible family of curves, which we make up from the system of q-families which are admissible on the continuous segments of $x_s(t)$. For zero values of the parameters the extremal is contained in the $(p+1)$-parameter family constructed in this manner. For this it is necessary to fulfill the stationarity condition

$$\Delta I = 0 \tag{2.1}$$

$$I = \psi + \sum_{i=0}^{q-1} \int_{t_i^+}^{t_{i+1}^-} \Big(\sum_{s=1}^{n} \lambda_s x_s^{\cdot} - H \Big) dt$$

$$\psi = \psi_0 + \sum_{l=1}^{p} \rho_l \psi_l, \qquad H = \sum_{s=0}^{n} \lambda_s f_s + \sum_{j=1}^{m} \mu_j \varphi_j \quad (\lambda_0 = -1)$$

Here ρ_l, $\lambda_s(t)$, $\mu_j(t)$ are undetermined multipliers. Applying Lagrange's lemma, from (2.1) we obtain the following relations:

1) on the segments of continuity of the phase coordinates

$$\lambda_s^{\cdot} + \frac{\partial H}{\partial x_s} = 0 \ \ (s = 1, \ldots, n), \qquad \frac{\partial H}{\partial u_k} = 0 \quad (k = 1, \ldots, r) \tag{2.2}$$

2) at the instants $t_i^-, t_i^+ \quad (i = 0, 1, \ldots, q)$

$$\frac{\partial \psi}{\partial x_s(t_0)} = 0, \qquad \frac{\partial \psi}{\partial t_0} = 0, \qquad \frac{\partial \psi}{\partial x_s(T)} = 0, \qquad \frac{\partial \psi}{\partial T} = 0$$

$$\lambda_s^+(t_i) - \frac{\partial \psi}{\partial x_s^+(t_i)} = 0, \qquad (H^+)_{t_i} = -\frac{\partial \psi}{\partial t_i^+} \qquad (i = 0, 1, \ldots, q-1) \tag{2.3}$$

$$\lambda_s^-(t_i) + \frac{\partial \psi}{\partial x_s^-(t_i)} = 0, \qquad (H^-)_{t_i} = \frac{\partial \psi}{\partial t_i^-} \qquad (i = 1, \ldots, q)$$

$$(s = 1, \ldots, n)$$

The discontinuities in the equations do not affect the continuity of the multipliers $\lambda_s(t)$ and of the functions H [1].

To determine the $(2n + r + m)$ unknown functions $x_s(t)$, $\lambda_s(t)$, $u_k(t)$, $\mu_j(t)$ and the $[2(n+1)(q+1) + p]$ unknown constants we have the $(2n + r + m)$ Eqs. (1.1), (2.2) and the $[2(n+1)(q+1) + p]$ relations (1.2), (2.3).

In problems for systems with discontinuities only in the phase coordinates, among the functions ψ_l are included the equalities

$$\psi_i = t_i^- - t_i^+ = 0 \qquad (i = 0, 1, \ldots, q < p)$$

while for systems with continuous phase coordinates or with a fixed magnitude of discontinuity, also the following equalities

$$\psi_\nu = x_s^-(t_i) - x_s^+(t_i) = X_{si} \qquad (X_{si} = 0 \ \ \text{or const}, \ \nu = 1, \ldots, n(q+1))$$

Here, the known conditions obtained in [1 – 3] follow from (2.3).

The Weierstrass inequality which serves as the necessary condition for the strong minimum of a functional, reduces to the form

$$H(x, \lambda, u, \mu, t) \geqslant H(x, \lambda, u^*, \mu, t) \tag{2.4}$$

where $u(t)$ are the controls yielding the minimum of J, while $u^*(t)$ are any admissible controls. Condition (2.4) should be fulfilled on the continuity segments (t_i^+, t_{i+1}^-) $(i = 0, 1, \ldots, q-1)$ of the phase coordinates.

3. **Example.** Let us consider the time-optimality problem for the system

$$x_1^{\cdot} = x_2, \qquad x_2^{\cdot} = u \qquad (|u| \leqslant 1) \tag{3.1}$$

These equations describe, for example, the motion of a point of unit mass under the action of a control of bounded magnitude. If we treat this motion as the motion of a pellet in a trough with stops at the ends, then the range of variation of coordinate x_1 is bounded

$$|x_1| \leqslant X_1 \tag{3.2}$$

At the instant $t = t_1$ when the point reaches a stop ($x_2^-(t_1) > 0$ for the right-hand stop or $x_2^-(t_1) < 0$ for the left-hand one) an impact takes place. Assuming the impact to be inelastic, we get that the velocity abruptly jumps to zero,

$$x_2^+(t_1) = 0 \tag{3.3}$$

When the stop is reached, the following relations hold for the phase coordinate x_1 and for t

$$|x_1^-(t_1)| - X_1 = 0, \qquad x_1^+(t_1) - x_1^-(t_1) = 0, \qquad t_1^+ - t_1^- = 0 \tag{3.4}$$

For system (3.1) under the constraint (3.2) leading it to (3.3), (3.4), it is necessary to choose u such that the transfer time from the state $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}$ to the equilibrium position $x_1(T) = x_2(T) = 0$ is minimal.

We consider the influence on the process to be synthesized only of the right-hand bound, since all the conditions are obtained analogously for the left-hand bound. We set up the functions H and ψ in the form

$$H = -1 + \lambda_1 x_2 + \lambda_2 u \tag{3.5}$$

$$\psi = \rho_1 x_2^+(t_1) + \rho_2 [x_1^-(t_1) - X_1] + \rho_3 [x_1^+(t_1) - x_1^-(t_1)] + \rho_4 (t_1^+ - t_1^-) \tag{3.6}$$

Here, in the function ψ we have omitted terms relating to the instants t_0, T, since the values of the phase coordinates at these instants have been fixed.

Conditions (2.2), (2.4) lead to the equations

$$\lambda_1^{\cdot} = 0, \qquad \lambda_2^{\cdot} = -\lambda_1, \qquad u = \operatorname{sign} \lambda_2 \tag{3.7}$$

From conditions (2.3) we have

$$\lambda_2^-(t_1) = 0, \qquad (H^-)_{t_1} = (H^+)_{t_1} \tag{3.8}$$

Taking (3.3), (3.7) and $x_2^-(t_1) > 0$ into account, from $H^- = H^+$ we obtain $\lambda_1^-(t_1) > 0$ which corresponds to the necessity of fulfilling the condition $\lambda_2(t) > 0$ on the semi-interval $[t_0, t_1)$. Thus, the motion is optimal for $u = 1$. The subsequent motion on the semi-interval $(t_1^+, T]$, ensuring the optimal transfer of the point from the position $x_1 = X_1$, $x_2 = 0$ to the origin is without any peculiarity, with a switching of the control at the curve $ON, x_1 = \frac{1}{2} x_2^2$ (Fig. 1.)

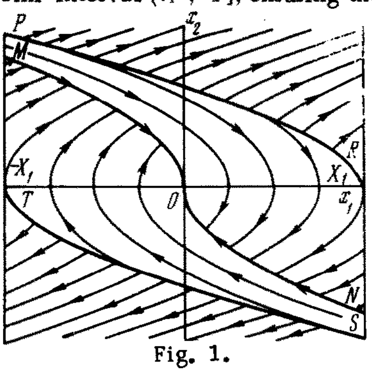

Fig. 1.

Thus, for an optimal motion with an impact on the boundary $x_1 = X_1$ the switching of the control takes place at $x_1 = X_1$ and on the curve ON. The region of initial states the optimal motion from which is achieved under two switchings of the control, lies above the curve PR (or, respectively, under the curve ST for motions reaching the left-hand boundary). For motions starting on the curves PR and ST the minimal transfer time to the origin can be obtained in two ways: with or without going onto the boundary. Using this property, for PR we obtain

$$x_2 + \sqrt{x_1 + {}^1/_2 x_2^2} - {}^1/_2 \sqrt{x_2^2 + 2(X_1 - x_1)} - \sqrt{X_1} = 0$$

while the curve ST is situated symmetrically relative to the origin. The family of optimal trajectories is depicted in Fig. 1.

The time-optimal motion of system (3.1) without constraints on the phase coordinates is achieved with two intervals of constancy of the control, $u = \pm 1$ [5]. The presence in the system of stops has led to the delineation of a region the optimal motion from which takes place with three intervals of constancy of the control. From points in the phase plane, lying in the indicated region (above PR and below ST), the transfer to the origin takes place in a lesser time than for systems without stops. This happens as a consequence of the fact that a constraint of the second type may not be violated [2] and serves as a component part of the system, changing its properties. Herein lies the principal difference between constraints of the second type and constraints of the first type imposed externally [2].

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